Mereologies

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Outline of Topics

I. Classical Mereology
II. Weaker Mereologies
III. Temporal Mereologies
IV. Mereotopologies
V. Containment Relations

Classical Mereology

Formal Theories

A formal theory is specified by a collection of formulas of a formal deductive system. These formulas are the axioms of the theory. Think of the axioms as postulates of the theory—the basic assumptions made for reasoning about a given topic.

For this class, we will generally assume that our formal deductive system is standard first-order with designated symbols for mereological (part-whole) relations.

Primitive vs. Defined Relations

- All formal mereologies assume one or more relation primitives. These relations are not defined. Instead, their logical properties are fixed by the axioms of the theory. All primitives must be interpreted in a way that preserves these logical properties.
- All other relations (the defined relations) in the mereology are defined in terms of the primitive relations.

Axioms, Definitions, and Theorems

In applying a theory, 3 different kinds of formulas are important:

- The axioms of a theory are the formulas which fix the logical properties of the primitive relations.
- The definitions of a theory are the formulas which introduce the defined relations.
- The theorems of a theory are the formulas that can be derived from the axioms and definitions using the machinery of the background deductive system (e.g. a set of axioms and deductive rules for standard first-order logic).

Models for Formal Theories

A model is given for a formal theory by specifying:

1. A domain (i.e., a set of individuals over which quantifiers range) for the model
2. Interpretations for the primitives of the theory as relations on the domain which satisfy the theory's axioms (I.e. interpretations of the primitives over the domain which make the axioms true).

Classical Mereology

Our version of classical mereology assumes 2 kinds of entities in the domain of quantification: individuals (x, y, z,…) and collections of individuals (A, B, C,…).

One mereological primitive: the binary relation P where on the intended interpretation Pxy means:

x is part of y

Classical Mereology: Definitions

PPxy =: Pxy & x $\neq$ y

(x is a proper part of y: x is part of y and x isn't identical to y)

Example: my hand is a proper part of my body

Oxy =: $\exists$z (Pzx & Pzy)

(x overlaps y: some object z is part of both x and y)

Example: my vertebral column overlaps my pelvis

DSxy =: ~ Oxy

(x and y are discrete: x and y do not overlap)

Example: my nose and my left foot are discrete

SUM(x, A) =: $\forall$y (y $\in$ A $\rightarrow$ Pyx) & $\forall$z (Pzx $\rightarrow$ $\exists$y (y $\in$ A & Ozy))

(x is the sum of the members of A: every member of A is part of x and every part of x overlaps some member of A)

Classical Mereology: Axioms

(A1) Pxx (reflexivity: for any x, x is part of itself)

(A2) Pxy & Pyz → Pxz (transitivity: if x is part of y and y is part of z, then x is part of z)

(A3) Pxy & Pyx → x = y (antisymmetry: if x is part of y and y is part of x, then x and y are identical)

Classical Mereology: Axioms

(A4) $\sim Pxy \rightarrow \exists z(Pzx \ \& \ DSzy)$

(the strong supplementation principle: if x is not part of y, then there is some z that is part of x and discrete from y)

(A5) $\exists y\ y \in A \rightarrow \exists x\ SUM(x, A)$

(Universal Fusion Principle: every non-empty collection of individuals has a sum)

Some Theorems of Classical Mereology

(T1) $\forall z\,(Ozx \rightarrow Ozy) \rightarrow Pxy$

(if everything that overlaps x also overlaps y, then x is part of y)

(T2) $\forall z\,(Ozx \leftrightarrow Ozy) \rightarrow x = y$

(if x and y overlap the same things, then x and y are identical)

(T3) $SUM(x, A)\ \&\ SUM(y, A) \rightarrow x = y$

(every collection of individuals has at most one sum)

A Class of Models for Classical Mereology

Let S be any non-empty set. Let SB(S) be the set of all non-empty subsets of S. Take SB(S) as the domain of the model. Interpret P as the subset relation on SB(S).

Exercises: (i) what interpretations will PP, O, DS, and SUM have in these models? (ii) show that axioms (A1)-(A5) are satisfied over these models.

A Specific Model

Let S = {1, 2}. List the members of SB(S) and the interpretations of P, PP, O, DS, and SUM. Note that the classical axioms are satisfied.

A Spatial Model

- Let S = $\Re^3$. SB($\Re^3$) is the set of all non-empty subsets of $\Re^3$. P is interpreted as the subset relation on SB($\Re^3$), PP is interpreted as the proper subset relation on SB($\Re^3$), O is interpreted as the relation which holds between subsets with a non-empty intersection, and DS is interpreted as the relation which holds between subsets with an empty intersection.

II. Weaker Mereologies

Some Unintuitive Commitments of Classical Mereology

$\exists y\ y \in A \rightarrow \exists x\ \mathrm{SUM}(x, A)$

For every non-empty collection of objects, there is some object which they all compose.

$\mathrm{SUM}(x, A)\ \&\ \mathrm{SUM}(y, A) \rightarrow x = y$

Every collection of objects composes at most one object.

Weaker Mereologies

Can be generated by just removing axioms from classical mereology:

(A1) Pxx

(A2) Pxy & Pyz → Pxz

(A3) Pxy & Pyx → x = y

(A4) ~Pxy → ∃z(Pzx & DSzy)

OR:

(A1) Pxx

(A2) Pxy & Pyz → Pxz

(A4) ~Pxy → ∃z(Pzx & DSzy)

OR problematic axioms can be replaced by weaker assumptions

Define Uyz =: $\exists$w (Pyw & Pzw)

(y and z underlap: there is some object w of which y and z are both parts)

(A1) Pxx

(A2) Pxy & Pyz $\rightarrow$ Pxz

(A3) Pxy & Pyx $\rightarrow$ x = y

(A4) ~Pxy $\rightarrow$ $\exists$z(Pzx & DSzy)

(A5*) $\exists$y y $\in$ A & $\forall$y$\forall$z(y,z $\in$ A $\rightarrow$Uyz) $\rightarrow$ $\exists$x SUM(x, A)

(every non-empty collection of pairwise underlapping objects has a sum)

Another weaker mereology

Define zDIFyx =:∀w (Owz ↔ ∃v (Pvx & DSvy & Ovw))

(z is the difference of y in x: z consists of all of x that is discrete from y)

(A1) Pxx

(A2) Pxy & Pyz → Pxz

(A3) Pxy & Pyx → x = y

(A4/5*) ~Pxy → ∃z zDIFyx

(if x is not part of y, then there is an object which is the difference of y in x)

Note: (A4/5*) is stronger than (A4) and weaker than (A5).

Exercise

- Which (possibly) undesirable results do the two preceding mereologies avoid?
- What might we do to avoid other (possibly) undesirable commitments?

III. Temporal Mereologies

A temporal version of classical mereology (TCM)

TCM has objects, collections of objects, and times as disjoint sorts of entities. Use variables: w, x,y,z for objects; A,B, C, for collections; and s,t for times.

TCM has one mereological primitive: the ternary relation P (parthood). P takes two objects and a time as its arguments.

$$Pxyt$$

Is intended as:

Object x is part of object y at time t.

Defined Relations

PPxyt =: Pxyt & ~Pyxt

(x is a proper part of y at t: x is part of y at t and y is not part of x at t)

Oxyt =: ∃z (Pzxt & Pzyt) (x and y overlap at t: some object z is part of both x and y at t)

DSxyt =: ~Oxyt (x and y are discrete at t: x and y do not overlap at t)

Ext =: Pxxt (x exists at time t: x is part of itself at t)

TCM's First Five Axioms

(AT1) $\exists t\, Ext$ (every object exists at some time)

(AT2) $Pxyt \rightarrow (Ext \,\&\, Eyt)$ (if x is part of y at t, then both x and y exist at t)

(AT3) $(Pxyt \,\&\, Pyzt) \rightarrow Pxzt$ (if x is part of y at t and y is part of z at t, then x is part of z at t)

(AT4) $(Pxyt \,\&\, Pyxt) \rightarrow x = y$ (if x is part of y at t and y is part of x at t, then x and y are identical)

(AT5) $(Ext \,\&\, {\sim}Pxyt) \rightarrow \exists z(Pzxt \,\&\, DSzyt)$ (if x exists at t and x is not part of y at t, then x has some part z at t that is discrete from y at t)

Temporal Relations for Collections

FP(A, t) =: $\exists x (x \in A)$ & $\forall x\ (x \in A \rightarrow \text{Ext})$

(A is fully-present at t: there is some member of A and all members of A exist at t)

SP(A, t) =: $\exists x\ (x \in A\ \&\ \text{Ext})$

(A is some-present at t: some member of A exists at t)

NP(A, t) =: $\forall x\ (x \in A \rightarrow {\sim}\text{Ext})$

(A is non-present at t: no members of A exist at t)

Summation Relations

Time-Dependent General Summation:
$SUM(z, A, t) =: Ezt \ \& \ \forall y \ (Oyzt \leftrightarrow \exists x(x \in A \ \& \ Oyxt))$
(z is a sum of the A's at t)

$SM_1(x, A) =: \forall t \ (Ext \rightarrow SUM(x, A, t))$
(whenever x exists, x is a sum of A's)
$SM_2(x, A) =: \forall t \ (SP(A, t) \rightarrow SUM(x, A, t))$
(whenever A is some-present, x is a sum of A's)
$SM_3(x, A) =: \forall t \ (FP(A, t) \rightarrow SUM(x, A, t))$
(whenever A is fully-present, x is a sum of A's)
$SM_4(x, A) =: \forall t \ ((Ext \vee SP(A, t)) \rightarrow SUM(x, A, t))$
(whenever x exists or A is some-present, x is a sum of A's)

Examples?

A Summation Axiom for TCM

(AT6) $SP(A, t) \rightarrow \exists z\ SUM(z, A, t)$

(if some member of A exists at t, then there is a sum of A's at t)

NOTE: We could have instead used one of the time-independent relations to generate a different temporal version of CM.

A Class of Models for TCM

Let S be any non-empty set. Let T be a non-empty set of times. Let $\wp(S)$ be the set of subsets of S. Let OB be any set of functions from T into $\wp(S)$ s.t. for any f, g $\in$ OB and any A $\subseteq$ OB:

1. there is some t, s.t. $f(t) \neq \varnothing$
2. if $f(t) = g(t)$, then $f = g$
3. if $f(t) \neq \varnothing$ and $f(t) \not\subseteq g(t)$, then there is some $h \in OB$ s.t. $h(t) \subseteq f(t)$ and $h(t) \cap g(t) = \varnothing$
4. if there is some $h \in A$ s.t. $h(t) \neq \varnothing$, then there is some $j \in OB$ s.t. $j(t) = \cup_{h \in A} h(t)$
5. if $f(t) \cap g(t) \neq \varnothing$, then there is some $h \in OB$ s.t $\varnothing \neq h(t) \subseteq f(t) \cap g(t)$

A Class of Models (con't)

If we interpret

P $\Rightarrow$ $\{<f, g, t>: f, g \in OB, t \in T, f(t) \neq \emptyset, \&$ $f(t) \subseteq g(t)\}$

then (AT1)-(AT6) are satisfied. (check!!)

What are the interpretations of the defined relations?

IV. Mereotopologies

Connection

We can add to any atemporal mereology the binary connection predicate C where Cxy is interpreted as:

x is connected to y

(i.e. x is zero distance from y)

(For a temporal theory, we would use a ternary predicate and Cxyt would be interpreted as:

x is connected to y at time t.)

Axioms for Connection (weaker version)

(AC1) Cxx

(connection is reflexive)

(AC2) $Cxy \rightarrow Cyx$

(connection is symmetric)

(AC3) $Pxy \rightarrow \forall z(Czx \rightarrow Czy)$

(if x is part of y, then every thing that is connected to x is also connected to y)

Additional Defined Relations

ECxy =: Cxy & DSxy

(x and y are externally connected: x and y are connected but do not overlap)

TPxy =: Pxy & ∃z(ECzx & ECzy)

(x is a tangential part of y: x is a part of y that is externally connected to something which is externally connected to y)

IPxy =: Pxy & ~ TPxy

(x is an interior part of y: x is part of y, but not a tangential part of y)

Additional Defined Relations

$SCx =: \forall y \forall z \; SUM(x, \{y, z\}) \rightarrow Cyz$

x is self-connected: any two parts that make up *all* of x are connected to each other (i.e. there is no way of dividing x into disconnected parts)

Comparing Mereotopologies

A weaker mereotopology uses 2 mereotopological primitives (P and C) and 8 axioms:

(A1) Pxx

(A2) Pxy & Pyz $\rightarrow$ Pxz

(A3) Pxy & Pyx $\rightarrow$ x = y

(A4) ~Pxy $\rightarrow$ $\exists$z(Pzx & DSzy)

(A5) $\exists$y y $\in$ A $\rightarrow$ $\exists$x SUM(x, A)

(AC1) Cxx

(AC2) Cxy $\rightarrow$ Cyx

(AC3) Pxy $\rightarrow$ $\forall$z(Czx $\rightarrow$ Czy)

A Stronger Mereotopology

Uses C (connection) as its only mereotopological primitive and defines:

$Pxy =: \forall z(Czx \rightarrow Czy)$

(x is part of y: everything that is connected to x is also connected to y)

5 Axioms:

(AC1) Cxx

(AC2) $Cxy \rightarrow Cyx$

(A3) $Pxy \;\&\; Pyx \rightarrow x = y$

(A4) $\sim Pxy \rightarrow \exists z(\, Pzx \;\&\; DSzy)$

(A5) $\exists y\, y \in A \rightarrow \exists x\, SUM(x, A)$

(dis)advantages of the Stronger Mereotopology

By defining parthood in terms of connection as

$Pxy =: \forall z(Czx \rightarrow Czy)$

We rule out cases in which an object has a point-sized, line-sized, or surface-sized gap in the middle

On the other hand, some ontologists want to rule out these sorts of irregularly-shaped objects anyway (along with lower-dimensional objects). If so, the stronger mereotopology may be preferable because it is more economical.

V. Containment Relations

Objects and Regions

We can expand a basic mereology by adding vocabulary for describing where objects are positioned in a fixed back ground space. To do this, we need to expand our domain of quantification to include immaterial spatial regions as well as the material objects (and immaterial holes or cavities) which are located at those regions.

To the formal vocabulary we add the unary region function r which takes object or region terms as arguments and where

$$r(x)$$

is interpreted as the spatial region at which x is exactly located.

Axioms for the Region Function

(AR1) $PPxy \rightarrow PPr(x)r(y)$

(if x is a proper part of y, then x's region is proper part of y's region)

(AR2) $r(r(x)) = r(x)$

(x's spatial region is its own spatial region)

Region Containment

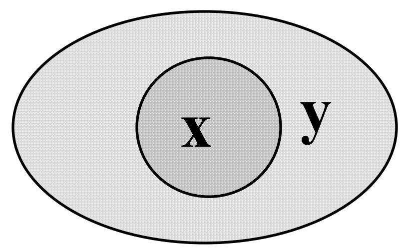
RCONxy =:Pr(x)r(y)

x is region-contained in y: x's region is part of y's region.

Examples:

My heart is region-contained

in my middle mediastinal space.



My larnyx is region-contained in my neck.

NOTE: The larnyx is part of the neck. But the heart is not part of the middle mediastinal space.

Convex Hulls

The convex hull of an object or region x is the smallest convex region in which x is region-contained.

We may add a separate primitive function mapping each object or region to its convex hull.

$$ch(x)$$

is x's convex hull.

Axioms for the convex hull function

(AR3) Pr(x)ch(x)

(x's region is part of x's convex hull)

(AR4) RCONxy $\rightarrow$ Pch(x)ch(y)

(if x is r-contained in y, then x's convex hull is part of y's convex hull)

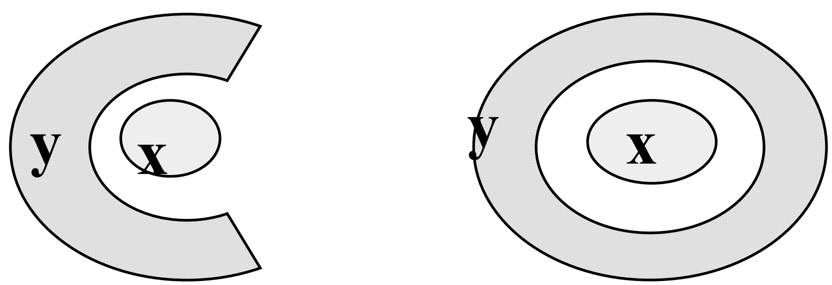
(AR5) ch(ch(x)) = ch(x)

(x's convex hull is its own convex hull)

Surrounding

SURxy =: Pr(x)ch(y) & DSr(x)r(y)

x is *surrounded* by *y*: x's region is part of y's convex hull but x's region does not overlap y's region.

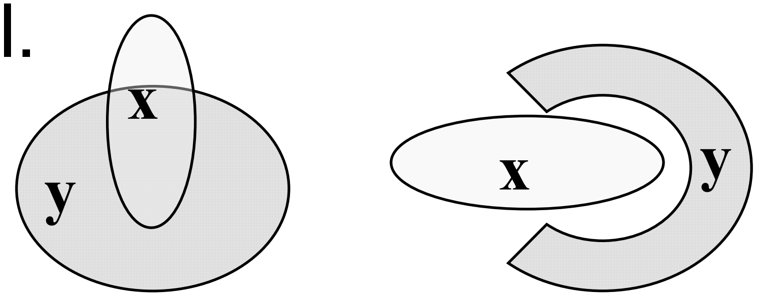


Examples: My pleural space is surrounded by my pleural membrane. A bolus of food in my stomach is surrounded by the wall of my stomach.

Partial Containment

PCONxy =: Or(x)ch(y)

x is *partially contained* in y: x's region overlaps y's convex hull.



Examples: My esophagus is partially contained in my thoracic cavity. My tooth is partially contained in its socket.

Logical Properties-- Transitivity

RCON is transitive. SUR and PCON are NOT transitive. Examples:

My heart is region-contained in my middle mediastinal space and my middle mediastinal space is region-contained in my thoracic cavity. So my heart is region-contained in my thoracic cavity.

A filling is partially contained in my tooth and my tooth is partially contained in its socket, but the filling is NOT partially contained in the socket.

Logical Properties-- Interaction with Parthood

The three defined containment relations interact differently with the parthood relation. For example:

If my heart is region-contained in my middle mediastinal space then any part of my heart is also region-contained in my middle mediastinal space.

BUT: My tooth is partially contained its socket even though some are its parts (e.g. its crown) are NOT partially contained in the socket.